

CONCERNING THE GEOMETRY OF ACYCLIC SETS

A Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS
UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY

BY

C. H. HARRY

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Introduction. In general, when sets of order relations are proposed for point collections the purpose is to construct linearly ordered arrays. One method of doing this is to take as undefined the concept of *betweenness*, where the condition "The point Y lies between the points X and Z " is symbolized by the triadic expression XYZ . The problem of the present paper is to formulate a set of axioms for this triadic expression which will lead to sets which are homeomorphic with subsets of general acyclic Peano spaces.* The work has been divided into two parts, the first contains an analysis of the space in terms of the betweenness relation alone, while the second deals with a discussion of the connection between the limit points and this betweenness relation.

PART I.

For the purpose of analysis the following five axioms will be employed:

AXIOM 0. ABC implies that A , B and C are distinct.

AXIOM 1. ABC implies CBA .

AXIOM 2. If A , B and C are distinct, then ABC or BCA or CAB is true.

AXIOM 3. ABC together with BCA implies CAB .

Definition. $\overline{UVW}$ means that UVW is true while VWU is false.

AXIOM 4. $\overline{AXB}$ together with either XCB or XBC implies $\overline{AXC}$.†

* A Peano space is one which is the continuous image of the unit interval. A necessary and sufficient condition for a set to be a Peano space is that it be metric, self compact, connected and locally connected. Cf. Hahn, *Wiener Berichte*, vol. 123 (1914), p. 2433 and Mazurkiewicz, *Fundamenta Mathematica*, vol. 1 (1920), p. 166. An acyclic Peano space is one which contains no simple closed curve.

† This collection differs from the assumptions used for linearly ordered arrays in Axiom 3. For linear sets this axiom would have to be replaced by a condition like:

AXIOM 3'. ABC implies the falsity of both BCA and CAB .

E. V. Huntington and J. R. Kline, *Transactions of the American Mathematical Society*, vol. 18 (1917), pp. 301-325, have listed twelve axioms which form the complete set of possible triadic relations among three and four points. These twelve axioms are either contained in or deducible from Axioms 0-2, 3', 4. Also, in this same paper, examples are given which, with very slight modifications, show Axioms 0-4 to be independent.

The choice of this particular set of axioms was influenced by the following properties of the cut points* of a connected space P . If a point Y separates two points X and Z in the space P they could be said to stand in the relations XYZ and ZYX and only these two. On the other hand, if no one of the three points X , Y and Z separates the other two they could be given all six possible betweenness relations. This definition satisfies Axioms 0-3 provided it is agreed to compare three points when and only when they are distinct. Since at most one of three points can separate the other two, then the relation $\overline{XYZ}$ will occur when and only when Y separates X and Z . To demonstrate Axiom 4 suppose X separates A and B . Then $P - X$ may be expressed as the sum of two sets M_a and M_b which have no point or limit point in common and which contain A and B respectively. If XCB or XBC is true, then C must lie in M_b , else $\overline{CXB}$ must hold. Hence, the point X separates A and C , i. e., $\overline{AXC}$ is true. Thus, all the properties deduced in Part I will be properties of the cut points of a connected set. In particular, cyclic chains and cyclic elements analogous to those of G. T. Whyburn† will be defined in terms of the betweenness relation; and, at the end of Part I, it will be shown that the above definition of betweenness for connected and locally connected sets makes the cyclic chains and elements defined by means of betweenness identical with those of the *Structure* paper. Thus, many of the properties of these chains and elements which are deduced by G. T. Whyburn may be shown without the use of connectivity.

Definition. By the *segment* AB is meant $A + B +$ all points X for which $\overline{AXB}$ is true. Also, in speaking of a segment AB it is always assumed that $A \neq B$.

Definition. By the *cyclic chain* $C(AB)$ is meant $A + B +$ all points X for which:

1° AXB is true, and 2° there is no point Y giving both $\overline{AYX}$ and $\overline{XYB}$, where again it is always assumed that $A \neq B$.

1. In this section properties of segments and cyclic chains will be deduced. The cyclic elements will be defined later.

* The point X is said to separate or cut a connected set P between two points A and B provided $P - X$ may be expressed as the sum of two sets M_a and M_b neither of which contains a point or limit point of the other and which contain A and B respectively. From the fact that $M_a + X$ is a connected subset of $P - B$ it follows that B does not separate P between A and X . Similarly, A does not separate P between B and X .

† "Concerning the structure of a continuous curve," *American Journal of Mathematics*, vol. 50 (1928), pp. 167-194. Hereafter this paper will be referred to as the *Structure* paper.

1.0 $\overline{ABC}$ implies $\overline{CBA}$.

Proof. ABC implies CBA , Axiom 1. If BAC were true, then CBA together with BAC would give ACB , Axiom 3. But ACB , by Axiom 1, implies BCA , contrary to the definition of $\overline{ABC}$. Hence, $\overline{CBA}$ holds.

1.01 $\overline{ABC}$ implies the falsity of CAB .

1.02 Given AXB , then either $\overline{AXB}$ is true or both XBA and BAX are true.

1.1 AB lies in $C(AB)$.

Proof. Suppose $X \in AB - A - B$.* Now $\overline{BXA}$ and XYA imply $\overline{BXY}$, by Axiom 4. Therefore, there is no point for which XYA and XYB could be true simultaneously. Hence, both conditions that X belong to $C(AB)$ are fulfilled.

1.2 If $D \in C(AB)$, then $C(AB) = C(AD) + C(DB)$.

Proof. The demonstration is accomplished by showing that $C(AB)$ contains $C(AD) + C(DB)$ and that $C(AD) + C(DB)$ contains $C(AB)$. Proceeding with the proof of the former, suppose $X \in C(AD) - A - D$. It will first be shown that AXB is true by proving the falsity of $\overline{ABX}$ and XAB . Now $\overline{BAX}$ and AXD , by Axiom 4, would imply $\overline{BAD}$, contrary to the condition that D lie in $C(AB)$. On the other hand, $\overline{XBA}$ and BDA , by the same axiom, imply $\overline{XBD}$, which, together with $\overline{ABX}$, contradicts the fact that $X \in C(AD)$. Since AXB must be true the only thing which would prevent X from lying in $C(AB)$ would be the existence of a point Y giving $\overline{AYX}$ and $\overline{XYB}$. The existence of such a point will now be shown to be impossible. If Y did exist it would be distinct from D , for $\overline{ADX}$ contradicts the condition that X lie in $C(AD)$. As Y , D and X are distinct, YXD or YDX or DYX is true. An application of Axiom 4 shows that $\overline{AYX}$ and $\overline{BYX}$, together with either YXD or YDX , imply both $\overline{AYD}$ and $\overline{DYB}$, contrary to the fact that $D \in C(AB)$. Thus, as both YXD and YDX are false, $\overline{DYX}$ is true. However, $\overline{AYX}$ and $\overline{XYD}$ prevent X from being in $C(AD)$. Thus, the point Y can not exist and $X \in C(AB)$. By symmetry $C(AB)$ also contains $C(DB)$.

It remains to show that $C(AB)$ lies in $C(AD) + C(DB)$. In case $\overline{ABD}$ is true $B \in AD$. Thus, by 1.1, $B \in C(AD)$, so that by the above paragraph $C(AB)$ lies in $C(AD)$. A similar argument in the case of $\overline{BAD}$ will show

* $X, Y, \dots \in N$ means that $X, Y, \dots$ are points of the set N .

that $C(AB) = C(BA)$ lies in $C(DB)$. If these two possibilities are excluded either $\overline{ADB}$ is true or all six relations hold.

Case 1. $\overline{ADB}$.

Suppose $X \in C(AB) - A - B - D$. If $\overline{XBD}$ were true, then $\overline{XBD}$, together with BDA , would give $\overline{XBA}$, contrary to the fact that $X \in C(AB)$. Since $\overline{XBD}$ is false, then either BXD or XBD must be true. Likewise, either XDA or AXD is true. If both AXD and DXB were false the relations would be $\overline{ADX}$ and $\overline{XDB}$, contrary to the fact that $X \in C(AB)$. Thus, it may be assumed without loss that AXD is true. Hence, from the definition, if X were not in $C(AD)$ there would be a point Y giving the relations $\overline{AYX}$ and $\overline{XYD}$. As $X \in C(AB)$, $\overline{ABX}$ is false, so that $B \neq Y$. Now $\overline{AYX}$ and $\overline{XYB}$ imply that X is not in $C(AB)$, so that $\overline{XYB}$ must be false, i. e., either YBX or BXY must be true. However, $\overline{DYX}$, together with either YBX or BXY , implies, by Axiom 4, $\overline{DYB}$. But $\overline{ADB}$ and DYB give $\overline{ADY}$, and this, together with $\overline{DYX}$, gives $\overline{ADX}$, contrary to AXD . Thus, Y can not exist and X must lie in $C(AD)$.

Case 2. ADB, DBA and BAD are all three true.

Again assume that X is a point of $C(AB) - A - B - D$. In a manner similar to that of Case 1, it can be shown that either AXD or DXB is true. Assume AXD . If X is not in $C(AD)$ there must be a point Y giving $\overline{AYX}$ and $\overline{XYD}$, where again $Y \neq B$. Now $\overline{XYD}$ and either YBD or YDB imply $\overline{XYB}$, which, together with $\overline{AYX}$, contradicts the fact that $X \in C(AB)$. Hence, as both YBD and YDB are false, $\overline{BYD}$ must hold, i. e., $Y \in DB$. As Y lies in BD , then by 1.1 it lies in $C(DB)$. Hence, $C(YB)$ lies in $C(DB)$. It will next be shown that X lies in $C(YB)$. It is first necessary to show $\overline{AYB}$. Now $\overline{AYX}$ and $\overline{XYB}$ would imply that X is not in $C(AB)$. But if $\overline{XYB}$ is false either YXB or YBX must be true, and either one of these relations, together with $\overline{AYX}$ gives, by Axiom 4, $\overline{AYB}$. Thus, as $Y \in AB$, $Y \in C(AB)$ and, by Case 1 above, $C(AB)$ lies in $C(AY) + C(YB)$. Since $\overline{AYX}$ prevents X from lying in $C(AY)$, then $X \in C(YB)$. That is, $X \in C(DB)$, for $C(YB)$ lies in $C(DB)$. The above proof of $\overline{AYB}$ gives the following corollaries:

1.21 If $D \in C(AB)$ and $Y \in AD - D$, then $Y \in AB$.

1.22 If $X \in AB - A - B$, then $C(AX) \cdot C(XB) = X$.

1.3 If A, B and C are distinct points then $C(AB)$ lies in $C(AC) + C(CB)$.

Proof. If $\overline{ABC}$, $\overline{BCA}$ or $\overline{CAB}$ is true the theorem is a direct consequence

of 1.1 and 1.2. If these three possibilities are excluded then all six order relations are true for A, B and C . It will first be shown that AXC is true for any point X of $C(AB)$ which is different from A, B and C . If $\overline{ACX}$ were true, then, by 1.21, C would be a point of AB , contrary to the fact that $\overline{ACB}$ is excluded. On the other hand, $\overline{XAC}$ and $\overline{ACB}$, by Axiom 4, give $\overline{XAB}$, contrary to the fact that $X \in C(AB)$. Thus, as $\overline{ACX}$ and $\overline{XAC}$ are both false, AXC must hold. If X is not an element of $C(AC)$, then there exists a point Y such that $\overline{AYX}$ and $\overline{XYC}$ are both true. By 1.21 $Y \in AB$, and, by 1.2, $C(AB)$ lies in $C(AY) + C(YB)$. Since $\overline{AYX}$ prevents X from lying in $C(AY)$, then $X \in C(YB)$. But if $X \in C(YB)$, YXB is true. By Axiom 4, $\overline{CYX}$ and YXB give $\overline{CYB}$, i. e., $Y \in C(CB)$. Thus, by 1.1 and 1.2, $C(YB)$ lies in $C(CB)$. Hence, X , a point of $C(YB)$, lies in $C(CB)$.

1.4 If $X, Y \in C(AB)$, where $X \neq Y$, then either $X \in C(AY)$ or $Y \in C(AX)$.

Proof. By Axiom 4, $\overline{XAY}$ and AYB give $\overline{XAB}$, contrary to the fact that $X \in C(AB)$. Thus, as $\overline{XAY}$ is false, either AXY or AYX is true. Suppose AXY . If X were not in $C(AY)$ there would be a point P giving $\overline{APY}$ and $\overline{XPY}$. By 1.3 the chain $C(AB)$ lies in $C(AP) + C(PB)$. As $\overline{APY}$ prevents Y from lying in $C(AP)$ it lies in $C(PB)$, i. e., PYB is true. But $\overline{XPY}$ and PYB give $\overline{XPB}$, which, together with $\overline{APX}$, contradicts the fact that $X \in C(AB)$. The assumption AYX makes Y lies in $C(AX)$. Also, if X, Y, A and B are not distinct the theorem is trivial.

1.41 If X and Y are distinct points of $C(AB)$, then $XY - X - Y$ lies in AB .

1.5 If X and Y are distinct elements of $AB - A$, then either $\overline{AXY}$ or $\overline{AYX}$ is true.

Proof. As the theorem is trivial if A, B, X and Y are not distinct, assume them to be different points. If YAX were true, then $\overline{BYA}$ and YAX would imply $\overline{BYX}$, while $\overline{BXA}$ and XAY would imply $\overline{BXY}$, a contradiction. Since XAY is impossible, then either $\overline{AXY}$ or $\overline{AYX}$ must hold.

1.51 If X, Y and Z are distinct points of $AB - A - B$, then $\overline{XYZ}$ or $\overline{YZX}$ or $\overline{ZXY}$ is true.

Proof. By 1.5 either $\overline{AXY}$ or $\overline{AYX}$ is true. Assume $\overline{AXY}$. Likewise, either $\overline{AZX}$ or $\overline{AXZ}$ holds. Now $\overline{YXA}$ and $\overline{XZA}$, by Axiom 4, give the desired result. Suppose, then, that the relation is $\overline{AXZ}$. By Axiom 4, $\overline{BYA}$

and YXA give $\overline{BYX}$, while $\overline{BZA}$ and ZXA give $\overline{BZX}$. Applying the results of 1.5 to the segment XB , it follows that either $\overline{XYZ}$ or $\overline{XZY}$ is true.

2. The notion of cyclic element will now be introduced.

Definition. Two points X and Y are said to be *conjugate* provided there is no point P giving the relation $\overline{XPY}$.

Definition. By a *cut point* X of the space is meant a point X for which there exists two points U and V giving the relation $\overline{UXV}$.

Definition. By a *cyclic element* $C(X)$ is meant any maximal set $C(X)$ which contains X and is such that every pair of its points are conjugate.

2.1 If $X \in C(AB) - AB$ and $D(X)$ is the set of all points of $C(AB)$ which are conjugate to X , then either X is a cut point of the space or $D(X) + X$ is a cyclic element.

Proof. The proof will be divided into three parts.

1° Every two points U and V of $D(X)$ are conjugate.

The assumption that U and V are not conjugate entails the existence of a point W giving $\overline{UWV}$. By definition U , V and X are distinct. Also, as $W \in AB$ by 1.41, $X \neq W$. Since U is conjugate to X , $\overline{UWX}$ is false, i. e., either WUX or WXU is true. But, by Axiom 4, $\overline{VWU}$, together with either WUX or WXU , implies $\overline{VWX}$, contrary to the fact that W is conjugate to U . Hence, U and V must be conjugate, i. e., all points of $D(X)$ lie in the same cyclic element $C(X)$.

2° If a point Q lies in $D(X) - X$, then $D(X) + X$ is a cyclic element.

Suppose P is a point conjugate to every point of $D(X) + X$. The supposition that $P \cdot C(AB) = 0$ leads to the following contradiction. Either APB must be false or there must exist a point Y giving $\overline{AYP}$ and $\overline{PYB}$. Under the assumption that APB is false it follows that either $\overline{BAP}$ or $\overline{ABP}$ must hold. The relation $\overline{AXB}$ follows from the fact that $X \in C(AB) - AB$. But $\overline{PAB}$ and $\overline{AXB}$ give $\overline{PAX}$, while $\overline{PBA}$ and $\overline{BXA}$ imply $\overline{PBX}$, both of which contradict the fact that P and X are conjugate. Since APB is true and it is supposed that P does not lie in $C(AB)$, then there is some point Y giving $\overline{AYP}$ and $\overline{PYB}$. As $X \neq Q$, either X or Q , say Q , is distinct from Y . Also, as P is not in $C(AB)$ and Q is, then $P \neq Q$. Thus, one of the three $\overline{PYQ}$, $\overline{PQY}$ or $\overline{QPY}$ must hold. Since $\overline{PYQ}$ is impossible, as P is conjugate

to Q , then either YQP or YPQ is true. But $\overline{AYP}$ and $\overline{BYP}$, together with either YQP or YPQ , imply $\overline{AYQ}$ and $\overline{QYB}$, contrary to the fact that $Q \in C(AB)$. Hence, Y does not exist and $P \in C(AB)$. But as P is conjugate to X , then $P \in D(X)$, i. e., $D(X) + X$ is maximal.

3° If $D(X) = 0$ and there is a cyclic element $C(X) \neq X$, then for any point P of $C(X) - X$ the relations $\overline{AXP}$ and $\overline{PXB}$ are valid.

Just as in the proof of 2° above the relation APB must hold. Thus, as $P \cdot C(AB) = 0$, there exists a point Y giving $\overline{AYP}$ and $\overline{PYB}$. If the point X were different from the point Y , then reasoning identical with the latter part of 2° above would lead to the contradiction that $X \cdot C(AB) = 0$, where X replaces the letter Q used above. Thus, if $D(X) = 0$ and X is not a cyclic element $C(X)$ it is a cut point.

2.2 A cyclic element has at most two points in common with any segment.

Proof. The theorem is a direct consequence of 1.51 and the definition of cyclic element.

2.3 If a cyclic element $C(X)$ has two distinct points U and V in common with a cyclic chain $C(AB)$, then $C(X)$ lies in $C(AB)$.

Proof. Assume P is a point of $C(X)$ not in $C(AB)$. First, APB is true, for either $\overline{PAB}$ or $\overline{PBA}$, together with the fact that U and V are distinct points of $C(AB)$, would contradict the condition that P be conjugate to both U and V . Thus, there is a point Y giving $\overline{AYP}$ and $\overline{PYB}$. As U and V are distinct points of $C(AB)$, one of them, say U , is different from P and Y . Since P is conjugate to U , then $\overline{UYP}$ is false, i. e., either YUP or YPU must hold. But $\overline{AYP}$ and $\overline{BYP}$, together with either YUP or YPU imply $\overline{AYU}$ and $\overline{BYU}$, contrary to the fact that $U \in C(AB)$. Hence, $P \in C(AB)$.

2.31 If a cyclic element has two points in common with AB it lies in $C(AB)$.

2.4 If $C(X)$ and $C(Y)$ are distinct cyclic elements they have at most one point P in common. Further, if U and V are any points of $C(X) - P$ and $C(Y) - P$ respectively, then $\overline{UPV}$ is true.

Proof. As $C(X) \neq C(Y)$, one contains a point not in the other. Suppose $Q \in C(Y) - C(X)$. Thus, for some point R of $C(X)$ there is a point S

giving $\overline{QSR}$. As every point A of $C(X) - S - R$ is conjugate to R , then $\overline{ASR}$ is false, i. e., either $\overline{SAR}$ or $\overline{SRA}$ is true. But $\overline{QSR}$, together with either $\overline{SAR}$ or $\overline{SRA}$, implies $\overline{QSA}$. Thus, for any point U of $C(X) - S$ the relation $\overline{QSU}$ is valid. If B is any point of $C(Y) - S - Q$, then a similar argument gives either $\overline{SQB}$ or $\overline{SBQ}$. But $\overline{USQ}$, together with either of these, implies $\overline{USB}$. Thus, for any point U of $C(X) - S$ and any point V of $C(Y) - S$ the relation $\overline{USV}$ is true. Hence, $C(X)$ and $C(Y)$ have either one point in common or no point in common according to whether or not S lies in both $C(X)$ and $C(Y)$.

2.41 *If X is not a cut point of the space it lies in one and only one cyclic element.*

2.42 *If X and Y are distinct points of $C(AB) - AB$ and X is conjugate to Y , then $D(X) + X = D(Y) + Y$.*

2.43 *If X and Y are distinct points of $C(AB) - AB$ and $D(X) + X$ is not identical with $D(Y) + Y$, then there is a point P of AB giving $\overline{XPY}$.*

2.44 *If X and Y are points of $C(AB) - AB$ and $D(X) + X \neq D(Y) + Y$, then these sets have at most one point P in common. Further P must belong to AB and give the relation $\overline{XPY}$.*

Definition. A segment UV is said to be the *minimum segment containing a set N* provided N lies in UV but in no segment XY which is a proper subset of UV .

Definition. The space M is said to have *Property C* provided that given any three distinct points A , X and Y for which $AX \cdot AY - A \neq 0$ there is a point P lying in $AX \cdot AY - A$ such that AP is the minimum segment containing $AX \cdot AY$.

Although nothing has been said about the limit points of the space and the relation between the ordering and the limit points, it can be shown (Cf. Part II) that Property *C* is not only a consequence of the assumption that every segment is closed and compact, but it is a definitely weaker condition. The next five theorems will be deduced under the assumption that the set M has Property *C* in addition to satisfying Axioms 0-4.

2.5 *If the space M has Property C, then any cyclic chain $C(AB)$ is identical with the set consisting of AB together with all cyclic elements having exactly two points in common with AB .*

Proof. It will be shown that for any point X of $C(AB) - AB$ the set $D(X)$ has exactly two points in common with AB . The remainder of the theorem will then follow from 2.1, 2.2 and 2.21. Let $N(A)$ be the set $AX \cdot AB$. Suppose first that $N(A) = A$. If X were not conjugate to A there would be a point P giving $\overline{APX}$. But, by 1.51, P must then be a point of $AB - A - B$. This is impossible as $N(A) = A$. Hence, A is conjugate to X , i. e., $A \in D(X)$. On the other hand, if $N(A) - A \neq 0$, then by Property C there is a point A' of $N(A) - A$ such that AA' is the minimum segment containing $N(A)$. As $A' \in AB$ and $X \in C(AB) - AB$, the points A' and X are distinct. Any point Y giving $\overline{A'YX}$ will, by 1.51, lie in both AX and AB , i. e., $Y \in N(A)$. But $\overline{AA'X}$ and $A'YX$ imply $\overline{AA'Y}$, contrary to the fact that Y , being a point of $N(A)$, must lie in AA' . Hence, as such a point Y can not exist, the points A' and X are conjugate, i. e., $A' \in D(X)$. Since $\overline{AA'X}$ is true and $X \in C(AB)$, $A' \neq B$. By symmetry either $B \in D(X)$ or there is a point B' distinct from A and B which lies in both $D(X)$ and AB . If either A' or B' fails to exist then it follows at once that the set $D(X)$ has two and only two points in common with AB . If A' and B' both exist, then $A' \neq B'$, for $\overline{AA'X}$ and $\overline{XB'B}$ are both true and $X \in C(AB)$.

2.6 If M has Property C and A, B and P are any three points for which APB is true but $P \cdot C(AB) = 0$, then there is a point Y of $C(AB)$ such that $C(AP)$ lies in $C(AB) + C(YP)$ while for any point X of $C(AB) - Y$ the relation $\overline{XYP}$ is true.

Proof. As APB is true but P does not belong to $C(AB)$, there is a point Z giving $\overline{AZP}$ and $\overline{PZB}$. Hence, as $PA \cdot PB - P \neq 0$, Property C may be applied. Let Y be that point of $PA \cdot PB - P$ such that PY is the minimum segment containing $PA \cdot PB$. Either $\overline{ABY}$ or $\overline{BAY}$, together with BYP and AYP , contradicts the condition APB . Hence, as both $\overline{ABY}$ and $\overline{BAY}$ are false, AYB must hold. If Y did not belong to $C(AB)$, then there would have to be a point W giving $\overline{AWY}$ and $\overline{YWB}$, for AYB is true. But, by 1.41, W lies in both AP and PB , i. e., $W \in AP \cdot PB - P$. However, $\overline{PYA}$ and YWA imply $\overline{PYW}$, contrary to the fact that PY must contain W . Hence, Y is a point of $C(AB)$. By 1.2, $C(AP) = C(PB)$ lies in $C(AY) + C(YP)$ and $C(AY)$ lies in $C(AB)$. Also, by the same theorem, $C(AB)$ lies in $C(AY) + C(YB)$. Thus, for any point X of $C(AB) - Y$, the relation $\overline{PYX}$ follows from $\overline{PYA}$ and $\overline{PYB}$.

2.61 Under the same hypothesis as 2.6, the chain $C(YP)$ has exactly the point Y in common with $C(AB)$.

2.62 If M has Property C and A, B and P are any three distinct points such that P does not lie in $C(AB)$, then there is a point Y of $C(AB)$ for which $C(AP)$ lies in $C(AB) + C(YP)$. Further, if $X \in C(AB) - Y$, then $\overline{PYX}$ is true. Finally, $C(YP)$ has exactly the point Y in common with $C(AB)$.

2.7 If (M_i) is any countably infinite collection of points M_i and M is a space having Property C , then there is a collection $(C(X_iP_i))$ of chains $C(X_iP_i)$ having the following properties:

$$1^0 \quad \sum_1^\infty C(X_iP_i) \text{ contains } \sum_1^\infty M_j;$$

$$2^0 \quad \text{For } k > 1, C(X_kP_k) \cdot \sum_1^{k-1} C(X_iP_i) = X_k; \text{ and}$$

3^0 If $r > s$, X any point of $C(X_rP_r) - X_r$ and Y any point of $C(X_sP_s) - X_s$, then $\overline{XX_rY}$ is true.

Proof. Let $X_1 = M_1$ and $P_1 = M_2$. Let P_2 be the first M_i not in $C(X_1P_1)$. Let P_3 be the first M_i not in $C(X_1P_1) + C(X_1P_2)$. Continue in this manner. In general, having chosen $P_1, \dots, P_{n-1}$, let P_n be the first M_i which is not a point of the set $C(X_1P_1) + \dots + C(X_1P_{n-1})$. Clearly $\sum_i C(X_1P_i)$ contains $\sum_i M_i$. Since 2.6 makes it possible to assume the existence of a point X_2 such that $C(X_1P_1)$ and $C(X_2P_2)$ have properties 2^0 and 3^0 , suppose that for $n-1 > 1$ the points $X_2, \dots, X_{n-1}$ have been so chosen that $C(X_1P_1), \dots, C(X_{n-1}P_{n-1})$ satisfy conditions 2^0 and 3^0 . Pick a point X^2 in $C(X_1P_1)$ such that $\overline{P_nX^2X}$ is true for every point X of $C(X_1P_1) - X^2$. Let k_2 be the first $i \leq n-1$ such that $\overline{P_nX^2Y_{k_2}}$ is false for some point Y_{k_2} of $C(X_{k_2}P_{k_2}) - X^2$. Now $X^2 = X_{k_2}$ as follows. Suppose, on the other hand, that X^2 and X_{k_2} are distinct. Since X^2 lies in a $C(X_iP_i)$ for which $i < k_2$, then by the construction of $C(X_{k_2}P_{k_2})$ the relation $\overline{P_{k_2}X_{k_2}X^2}$ holds. But X_{k_2} lies in some $C(X_jP_j)$ where $j < k_2$, so from the definition of k_2 the relation $\overline{P_nX^2X_{k_2}}$ must be valid. Applying Axiom 4, $\overline{P_{k_2}X_{k_2}X^2}$ and $X_{k_2}X^2P_n$ give $\overline{P_{k_2}X_{k_2}P_n}$, while $\overline{P_nX^2X_{k_2}}$ and $X^2X_{k_2}P_{k_2}$ imply $\overline{P_nX^2P_{k_2}}$. Thus, Y_{k_2} is distinct from P_{k_2} and X_{k_2} . That is, $P_{k_2}Y_{k_2}X_{k_2}$ must be true. However, $\overline{X^2X_{k_2}P_{k_2}}$ and $X_{k_2}Y_{k_2}P_{k_2}$ imply $\overline{X^2X_{k_2}Y_{k_2}}$, while $\overline{P_nX^2X_{k_2}}$ and $X^2X_{k_2}Y_{k_2}$ give $\overline{P_nX^2Y_{k_2}}$, contrary to hypothesis. Since X^2 and X_{k_2} are the same point, then 2.6 may be applied to find a point X^3 of $C(X_{k_2}P_{k_2}) - X_{k_2}$ such that $\overline{P_nX^3X}$ is true for each point X of $C(X_{k_2}P_{k_2}) - X^3$. From the choice of X_{k_2} the relation $\overline{X^3X_{k_2}Y}$ holds for each point Y of $\sum_1^{k_2-1} C(X_iP_i) - X_{k_2}$. Thus, for each point U of $\sum_1^{k_1} C(X_jP_j) - X^3$, $\overline{P_nX^3U}$ follows from $\overline{P_nX^3X_{k_2}}$ and Axiom 4.

If k_3 is the first $k < n$ such that some point Y_{k_3} of $C(X_{k_3}P_{k_3}) - X^3$ does not stand in the relation $\overline{P_n X^3 Y_{k_3}}$, then $k_3 > k_2$. Just as before, $X^3 = X_{k_3}$. Continuing in this manner there will eventually be a point X^m such that $\overline{P_n X^m X}$ is true for every X of $\sum_1^{n-1} C(X_i P_i) - X^m$, where $X^m \in \sum_1^{n-1} C(X_i P_i)$. Let $X_n = X^m$. It follows from 1.1-1.2 that $\sum_1^n C(X_i P_i) = \sum_1^n C(X_j P_j)$.

2.8 Axioms 0-2, 3', 4 imply Property C.*

Proof. Axiom 3' implies $\overline{UVW}$ whenever UVW is true, so that Property C is trivial.

2.9 Axioms 0-4 imply that every segment AB possesses a linear order.

Proof. Let the symbol $X \propto Y$ mean that " X precedes Y ." For distinct points X and Y of AB let $X \propto Y$ when:

1° $X = A$ or $B = Y$; and 2° When $\overline{AXY}$ is true.

Thus, by 1.6, if X and Y are distinct points of AB , then $X \propto Y$ or $Y \propto X$ but not both. If $X \propto Y$ and $Y \propto Z$, then $X \propto Z$ as follows. If $X = A$ or $Z = B$ the theorem is true immediately, so assume X and Z to be points of $AB - A - B$. As $X \propto Y$ and $Y \propto Z$, then Y also lies in $AZ - A - Z$. Now $\overline{ZYA}$ and YXA imply $\overline{ZYX}$, while $\overline{AXY}$ and XYZ give $\overline{AXZ}$, i. e., $X \propto Z$.

3. The notion of end point will now be introduced.

Definition. A point P is said to be an *end point* of the set M provided there exists a collection G of points of $M - P$ having the following properties:

1° $\overline{ZZ'P}$ or $\overline{Z'ZP}$ is true for any two distinct points Z and Z' of G ; and
2° If Q is any point of $M - P$, then some point Z of G gives $\overline{QZP}$.

Definition. A cyclic element is said to be *non-degenerate* provided it consists of more than one point.

3.1 If M has Property C, then M consists of the non-degenerate cyclic elements, the cut points and the end points.

Proof. Suppose X is neither a cut point nor contained in any cyclic element consisting of more than one point. Let Z be any point of $M - X$. It will be shown that the set $G = ZX - X$ has properties 1° and 2° listed above. Condition 1° is a consequence of 1.6. Let Q be any point of $M - X$.

Case 1. $Q \in C(ZX)$.

* Cf. footnote to p. 233.

As Q is not conjugate to X , for the cyclic element $C(X) = X$, there is a point P giving $\overline{QPX}$. By 1.1 and 1.21, the point P lies in $ZX - X$ and is therefore a point Z' of G .

Case 2. $Q \in M - C(ZX)$.

Since $\overline{ZXQ}$ is impossible as X is not a cut point of the space, then either ZQX or QZX is true. As $\overline{QZX}$ gives the desired result at once, assume $\overline{QZX}$ to be false. As $Q \in M - C(ZX)$, then by 1.1, QZX , QXZ and XQZ are all three true. Thus, from 2.6, there is a point Y lying in $C(ZX) - Z - X$ which gives the relation $\overline{QYZ'}$ for every point Z' of $ZX - Y$. From Case 1 it follows that some point Z' of $ZX - X$ stands in the relation $\overline{YZ'X}$. But $\overline{XZ'Y}$ and $Z'YQ$ imply $\overline{XZ'Q}$, so that $G = ZX - X$ is a set satisfying conditions 1° and 2° above. Therefore, X is an end point of the space.

3.2 If M is a connected and locally connected space in which betweenness has been defined in terms of separating point as was done on page 234, then Axioms 0-4 and Property C are a consequence of the definition.

Proof. It was seen earlier that $\overline{UVW}$ occurs when and only when V separates the points U and W in the set M . Thus, any segment AB consists of $A + B +$ all points X which separate A and B in M . G. T. Whyburn has shown* that AB is closed and compact in any connected and locally connected set. Suppose, then, that A , B and P are any three distinct points for which APB , ABP and BAP are all true. Assume further that $AP \cdot AB - A \neq \emptyset$. Let $N = AP \cdot AB$ and order AB in the manner of 2.9. As AB and AP are closed and compact sets, then N has a last point K which lies in N . From the hypothesis K is distinct from A , B and P . If X is any point giving $\overline{AXP}$ and $\overline{AXB}$ then X lies in N . Since K is the last point of N in AB , either $X = K$ or $\overline{AXK}$ is true. Thus, AK is the minimum segment containing N .

3.3 If M is any connected and locally connected space in which betweenness has been defined in terms of separating point as above, then the cyclic chains, non-degenerate cyclic elements, cut points and end points defined by means of the betweenness relations are identical with those defined in the Structure paper.

Proof. In the Structure paper a non-degenerate cyclic element was characterized as being a maximal set $C(X)$ such that no two points U and V of $C(X)$ are separated in M by any point. Since $\overline{UXV}$ means X separates U

* "On the structure of connected and connected im kleinen point sets," *Transactions of the American Mathematical Society*, vol. 32 (1929), no. 4, p. 927.

and V , then the non-degenerate cyclic elements are the same. The cyclic chains of the *Structure* paper are shown to be identical with the set AB together with all cyclic elements which have exactly two points in common with AB . Hence, by 2.5, the cyclic chains defined by means of the betweenness relations are identical with those of G. T. Whyburn. The definition of end point in the *Structure* paper is the following: "The point P of a continuum M is an end point of M provided that if N is any subcontinuum of M containing P , then P is not a limit point of any connected subset of $M - N$." An equivalent definition for locally connected continua is the following: * P is an end point when and only when there exists a countable collection of open sets O_i such that P is the only point common to every O_i , the boundary of each O_i is exactly one point Z_i , and O_i contains $O_{i+1} + Z_{i+1}$ for every i . Since Z_{i+1} separates Z_i and P for every i , then the definition of end point in terms of betweenness gives the same points as those defined by Kuratowski and Whyburn.

PART II.

In this part it will be assumed that the set M is a separable metric space. The number $\rho(X, Y)$, which has the usual metric properties, will be used to denote the distance from X to Y . By the *diameter* $\delta(K)$ of a set K is meant the least upper bound of the numbers $\rho(X, Y)$ for all pairs of points X and Y of K . A set of betweenness axioms will now be formulated which will make M homeomorphic with a subset of an acyclic Peano space. In addition to Axioms 0-4 it will be supposed that M has been so ordered that the following four axioms are also true:

AXIOM 5. If A is a limit point of a set N there is no point P such that $\overline{APX}$ is true for every point X of N different from A .

AXIOM 6. M has Property C.†

AXIOM 7. If $(C(X_i Y_i))$ is a collection of cyclic chains $C(X_i Y_i)$ and P a point such that:

1° For each $k > 1$ the set $C(X_k Y_k) \cdot \sum_1^{k-1} C(X_i Y_i)$ is X_k or zero, and

2° For every number $\varepsilon > 0$ all but a finite number of the sets $C(X_i Y_i)$ have a point as close to P as ε , then the numbers $\delta(C(X_i Y_i))$ converge to zero with $1/i$.

* Cf. C. Kuratowski and G. T. Whyburn, "Sur les elements cyclique et leurs applications," *Fundamenta Mathematicae*, vol. 16 (1930), p. 307.

† Cf. p. 240.

AXIOM 8. If U and V are any two points of a cyclic element $C(X)$, then it is possible to send $C(X)$ into a subset of the unit interval by a homeomorphism T for which $T(U)$ and $T(V)$ are the ends of the interval.

The connection between the betweenness relations and the limit points is given by Axiom 5. However, it is to be noted that Axiom 5 is only a necessary condition for a point to be a limit point of a set.

Axiom 7 is used to prevent oscillation. For example, if the space M consists of the graph of $y = \sin 1/x$ for $1 \geq x > 0$ together with the origin, then M is connected. If betweenness is defined in M in terms of cut point as was done earlier, then Axioms 0-6 and 8 are true. However, such a set is not homeomorphic with the unit interval. It is evident that this example is not locally connected. Also, in the previously mentioned paper by C. Kuratowski and G. T. Whyburn,* it is shown for compact, connected and locally connected metric spaces that any collection of cyclic chains satisfying condition 1° of Axiom 7 is such that the diameters converge to zero. Thus, Axiom 7 stands as a substitute for the stronger property of local connectivity.

Up to the present time no assumption has been made concerning the cyclic elements. Many spaces, for example, any Peano space, may be so ordered that Axioms 0-7 are valid while Axiom 8 might or might not be true. In general, the definition of betweenness in terms of cut point for connected sets will not lead to Axiom 8 unless the cyclic elements are all degenerate. However, to restrict M to a space which is homeomorphic with a subset of an acyclic Peano space it is necessary to restrict the cyclic elements. Axiom 8, which at first sight seems to assume part of the desired result, is really the weakest condition that can be imposed.

4.0 Axioms 0-5 imply that the cyclic chains and cyclic elements are closed sets.

Proof. The cyclic elements will be treated first. Let P be a point which does not lie in the cyclic element $C(X)$. Thus, there is a point R of $C(X)$ and some point S of M such that $\overline{PSR}$ is true. If Y is any other point of $C(X)$ either SYR or SRY must be true, for $\overline{RSY}$ contradicts the condition that R and Y be conjugate. However, $\overline{PSR}$, together with either SRY or SYR , implies $\overline{PSY}$. Thus, for any point Y of $C(X) - S$, the relation $\overline{PSY}$ holds. Hence, by Axiom 5, P is not a limit point of $C(X)$. Therefore, each cyclic element is a closed set.

Suppose next that P is not a point of the cyclic chain $C(AB)$. Thus, by definition, either $\overline{PAB}$ or $\overline{PBA}$ is true or there exists a point Q giving

* *Loc. cit.*

$\overline{AQP}$ and $\overline{PQB}$. Since $C(AB)$ would lie in $C(AQ) + C(QB)$ if Q existed, then in any of the possibilities there would always, by Axiom 4, be a point Z giving $\overline{PZX}$ for every X of $C(AB) - Z$. Thus, by Axiom 5, $C(AB)$ is closed.

That each segment AB is closed does not follow from Axioms 0-5. However, it is a consequence of Axioms 0-6, as follows. If P is a limit point of AB , then by Axioms 0-5 both $\overline{ABP}$ and $\overline{BAP}$ are false. Hence, APB must be true. By 1.1 and 1.3 the set AB lies in $AP + PB$, where it is assumed that P is distinct from A and B . As P must then be a limit point of either AP or PB , assume it to be a limit point of AP . Thus, $AP - P$ is not zero. Now $AP - P$ lies in AB as follows. If $X \in AP - A - P$ the relation $\overline{AXP}$ is true by definition. But, by Axiom 5, for some point Y of AB the relation $\overline{YXP}$ is false, where Y can be chosen distinct from A, B, P and X . But $\overline{AXP}$ and either XPY or XYP give $\overline{AXY}$, so that by 1.1 and 1.5 $X \in AB$. Since $AP - P$ lies in AB and is different from zero, then Axiom 6 may be applied to find a point Z of $AB \cdot AP$ such that AZ contains $AP - P$. If Z and P were not the same point then $\overline{XZP}$ would be true for every point X of $AP - X$, contrary to Axiom 5. Thus, $Z = P$ lies in AB . The following theorem can now be stated.

4.1 If M satisfies Axioms 0-6, then each segment is a closed set.

4.11 If M satisfies Axioms 0-5, then any point P of a segment AB which is a limit point of the segment UV , where $U, V \in AB$ lies in UV .

4.2 If M satisfies Axioms 0-5, 7, then for each AB there is a homeomorphism T which sends AB into a subset of the unit interval. Further, A and B are transformed into the ends of the interval and T preserves the betweenness relations.

Proof. For convenience in notation the segment will be ordered in the manner given in 2.9. The proof will then be stated in terms of the symbol " α ". The demonstration follows along lines for similar proofs, the only difference occurring in the following situation. A point P may not be a limit point of both the set of points before P and the set of points after it. In this case the transformation must be so defined that open spaces are left in the unit interval to the right and left of the transform of P . For a separable metric space which has been linearly ordered in such a manner that the order preserves the limit points there are at most a countable number of points which are not limit points of both the points preceding and following them.* Hence, in AB there is a set $G = \sum_i P_i$ which is everywhere dense in AB and which

* Zarankiewicz, *Fundamenta Mathematicae*, vol. 12 (1928), p. 119.

contains every point P of AB which is not a limit point of both AP and PB . Suppose further that $P_1 = A$ and $P_2 = B$. If P_1 is not a limit point of P_1P_2 drop from the interval $(0, 1)$ the open interval $(0, 1/8)$. If P_2 is not a limit point of P_1P_2 drop the open interval $(1 - 1/8, 1)$. Call the resulting set K_1 and define $T(P_1)$ as $p_1^1 = 0$ and $T(P_2)$ as $p_2^1 = 1$. Between p_1^1 and p_2^1 there is one and only one interval belonging to K_1 . Let P_3 , if it exists, correspond to the midpoint p_1^1 of this interval. If P_3 is not a limit point of P_1P_3 drop from K_1 the open interval $(p_1^1 - 1/3^3, p_1^1)$. If P_3 is not a limit point of P_3P_1 drop from K_1 the open interval $(p_1^1, p_1^1 + 1/3^3)$. Let the resulting set be K_2 and relabel the points p_1^1, p_2^1, p_1^1 from left to right as p_1^2, p_2^2, p_3^2 . Continue in this manner. In general, having chosen $p_1^n, \dots, p_{v_n}^n$ and the set K_n , consider the pairs p_i^n and p_{i+1}^n , where $i = 1, 2, \dots, v_n - 1$. Suppose, for each k , that P_k^n is the point of G which corresponds to p_k^n . Let P_{m_i} , provided it exists, be the first P_m of G which is such that $P_i^n \propto P_{m_i}$ and $P_{m_i} \propto P_{i+1}^n$. Between p_i^n and p_{i+1}^n there is one and only one interval of K_n . If $p_{i'}^n$ designates the midpoint of this interval, then let $T(P_{m_i}) = p_{i'}^n$. If P_{m_i} is not a limit point of $P_1P_{m_i}$ drop from K_n the open interval $(p_{i'}^n - 1/(1+n)^3, p_{i'}^n)$; and if P_{m_i} is not a limit point of $P_{m_i}P_2$ drop from K_n the open interval $(p_{i'}^n, p_{i'}^n + 1/(1+n)^3)$. Do this for each i and let the remaining set of points of K_n be K_{n+1} . Label the points $\sum_i p_i^n$ together with the set $\sum_i p_{i'}^n$ from left to right as $\sum_{i=1}^{i=v_{n+1}} p_i^{n+1}$.

The proof that T is a homeomorphism which preserves the order and the proof that T can be extended to the whole set AB are rather tedious. Since they follow the lines of similar demonstrations it has been decided for brevity to omit them.

4.3 *Axioms 0-6 imply that there are at most a countable number of cyclic elements in any cyclic chain.*

Proof. Let the cyclic chain be $C(AB)$. Since any two cyclic elements $C(X)$ and $C(X')$ of $C(AB)$ have at most one point in common if they are distinct, then the supposition that there are an uncountable number of cyclic elements in $C(AB)$ implies the existence of an uncountable set G which lies in $C(AB) - AB$ and is such that distinct points X and X' of G lie in cyclic elements $C(X)$ and $C(X')$ which are distinct. By 2.5 each of these cyclic elements has exactly two points U and V in common with AB . Suppose U and V are so named that $U \propto V$. It is well known that any infinite subset of a linearly ordered array contains an infinite set which is monotonic with respect to this order. Since no three cyclic elements can have a point in

common, then for any infinite subset H of G there is an infinite subset H_0 of H having the following property:

If X and X' are any distinct points of H_0 and U and U' the corresponding points then either U always precedes U' or U' always precedes U .

In particular, since M is separable, some point X^* of G must be a limit point of a subset H of G . Without loss it may be assumed that it is possible to choose the set H_0 such that $H_0 = \sum_1^\infty X_i$ and the corresponding set $\sum_1^\infty U_i$ has the property that $U_i \propto U_{i+1}$. As $U_i \propto V_i$, then $U_i V_i \cdot U_{i+t+1} V_{i+t+1}$ is at most U_{i+t+1} if $t \geq 0$. As $C(X_k)$ lies in $C(U_k V_k)$, then it follows from the definition of the symbol " $\propto$ " that the collection of chains $C(U_i V_i)$ not only contains the set H_0 but satisfies the conditions of Axiom 7. Hence, their diameters converge to zero. Thus, X^* is a limit point of AB . By 4.0, X^* must then be a point of AB , contrary to the fact that X^* lies in $C(AB) - AB$. Hence, the collection of cyclic elements belonging to $C(AB)$ is countable. This same theorem may be proved by means of Axioms 0-5, 7 and the assumption that the segments AB are closed. However, the demonstration is much longer than the above. Using this theorem together with Axiom 8 it is possible to state the following:

4.4 *Axioms 0-8 imply that any $C(AB)$ may be sent into a subset of the unit interval by a homeomorphism S which sends A and B into the end points, preserves the order of AB and is such that the relation $S(X)S(Y)S(Z)$ on the interval implies XYZ .*

Proof. Let X be any point of $C(AB) - AB$ and let the elements $C(X)$ have the points U and V in common with AB . As $UV = U + V$, then the transformation T of 4.2 was such that no point W of AB was transformed by T into a point $T(W)$ lying between $T(U)$ and $T(V)$. By 4.3 the cyclic elements C of $C(AB)$ may be labeled $C_1, C_2, \dots$. Let U_i and V_i denote the points which C_i has in common with AB . By Axiom 8 there is a homeomorphism T_i sending C_i into a subset of $(T(U_i), T(V_i))$ in such a manner that $T(U_i) = T_i(U_i)$ and $T(V_i) = T_i(V_i)$. Thus, the transformation S defined as follows:

$$S(P) = T(P) \text{ if } P \in AB \text{ and } S(P) = T_i(P) \text{ if } P \in C_i,$$

gives a 1 — 1 correspondence between $C(AB)$ and a subset of the unit interval. That S is a homeomorphism with the desired properties follows from 4.0 and Axiom 7. Again, this theorem may be demonstrated by means of Axioms 0-5, 7 provided it is assumed that AB is closed.

The following theorem on the approximation of the space by means of cyclic chains is a direct consequence of 2.7 and the condition that M be separable.

4.5 Axioms 0-7 imply the existence of a collection of cyclic chains $R_i = C(X_i P_i)$ for which

1° $X_i \in X_1 P_i$; 2° Every point of M is a point or limit point of the set $\sum_i R_i$; and 3° $R_k \cdot \sum_{i=1}^{k-1} R_i = X_k$.

This theorem is analogous to the one announced in the previously mentioned paper by C. Kuratowski and G. T. Whyburn.* Such an approximation of the space will be used in constructing a homeomorphism between M and a subset of an acyclic Peano space, the method being to send the sets R_i into subsets of appropriate arcs in the acyclic Peano space.

5. A universal acyclic Peano space is such that every acyclic Peano space is homeomorphic with a subset of it. If (t_i) is an infinite collection of arcs t_i of an acyclic space, where $t_i \cdot t_j$ is a fixed point x , $i \neq j$, then x is called an infinite branch point of the space. T. Ważewski has shown † that an acyclic Peano space will be a universal acyclic space P provided that on each arc of P there is an everywhere dense subset of points which are infinite branch points of P . In fact, such an acyclic space has been shown to exist in the plane. Such a universal space will be employed in the demonstration of the following theorem.

5.1 If M is a separable metric space the points of which have been so ordered that Axioms 0-8 are valid, then M is homeomorphic with a subset of an acyclic Peano space.

Proof. Choose in M a collection of cyclic chains R_i having the properties announced in 4.5. Let P_0 be any universal acyclic Peano space and x_1 and p_1 any two infinite branch points of P_0 . Let r_1 be the arc from p_1 to x_1 and F_1 the set of infinite branch points which lie on r_1 . Let T_1 be the following homeomorphism:

1° T_1 sends the set $R_1 = C(X_1 P_1)$ into r_1 in such a manner that $T_1(X_1) = x_1$ and $T_1(P_1) = p_1$; and

2° T_1 transforms $R_i \cdot R_1$, where $i \neq 1$, into a point of F_1 . Theorem 4.4

* Loc. cit.

† *Annales Société Polonaise Mathematicae Cracovie*, vol. 2 (1923), p. 49; also, see Menger, *Fundamenta Mathematicae*, vol. 10, p. 108 and Gehman, *Transactions of the American Mathematical Society*, vol. 29 (1927), no. 3, p. 553.

gives a transformation T_{01} satisfying condition 1°, while a theorem due to Fréchet and Urysohn* states that T_{01} may be deformed into a transformation T_1 satisfying both 1° and 2°. Suppose that $T_1(X_2) = x_2$. Since x_2 is an infinite branch point it is possible to find an arc $x_2p'_2$ having just x_2 in common with r_1 . Choose on this arc a point p_2 different from x_2 which is an infinite branch point and let r_2 be the arc from x_2 to p_2 . If the set of infinite branch points lying on r_2 is F_2 , then let T_2 be a homeomorphism sending $R_2 = C(X_2P_2)$ into a subset of r_2 in such a manner that $T_2(X_2) = x_2$, $T_2(P_2) = p_2$ and $T_2(R_2 \cdot R_i)$ is a point of F_2 for each $i \neq 2$. Continue in this manner. In general, having chosen $T_1, \dots, T_{n-1}$ and the arcs $r_1, \dots, r_{n-1}$ in such a manner that $r_k \cdot \sum_1^{k-1} r_i = x_k$, let R_n be the cyclic chain $C(X_nP_n)$. Now X_n may lie in more than one of the sets R_i , where $n > i$. However, if it is assumed that the collection (R_i) was constructed as was done in the proof of 2.7 (Part I), it follows that if X_n lies in R_i and R_j , $n > i > j$, the points X_n , X_i and X_j are identical. Hence, for every i for which $T_i(X_n)$ has a meaning the point $T_i(X_n) = x_n$ is the same. Thus, as X_n lies in $\sum_1^{n-1} R_i$ there is some point x_n to which X_n corresponds, and only one such point. Let p'_n be a point for which the arc $x_np'_n$ has just the point x_n in common with $\sum_1^{n-1} r_i$. On this arc choose an infinite branch point p_n different from x_n and let r_n be the arc from x_n to p_n . If F_n is the set of infinite branch points lying on r_n , then let T_n be a homeomorphism which sends R_n into a subset of r_n in such a manner that $T_n(X_n) = x_n$, $T_n(P_n) = p_n$, and $T_n(R_n \cdot R_i)$ is a point of F_n , where $i \neq n$.

For a point X of $\sum_i R_i$ define $T(X)$ as $T_n(X)$ if $X \in R_n$. It follows at once that T is a 1—1 correspondence. In order to extend T to the limit points of $\sum_i R_i$ which do not lie in $\sum_i R_i$ use has been made of the following statement:

For each point Z of $M - \sum_i R_i$ there is one and only one collection (R_{t_i}) of sets R_i such that: 1° $t_1 = 1$; 2° $t_i \geq i$; 3° R_{t_i} and R_{t_j} have no point in common if $|i - j| > 1$; 4° R_{t_i} and $R_{t_{i+1}}$ have exactly the point $X_{t_{i+1}}$ in common; and 5° Z is a sequential limit point of $\sum_i X_{t_i}$. Thus, from the construction of T the corresponding arcs r_{t_i} would have properties 1°-4°. As the space P_0 is compact and locally connected, then there is one and only one point z which is a limit point of $\sum_i r_{t_i}$. Since the space is acyclic the

* M. Fréchet, *Mathematische Annalen*, vol. 68 (1910), p. 159 and P. Urysohn, *Fundamenta Mathematicae*, vol. 7 (1925), p. 83.

point z does not lie in $\sum_i r_{t_i}$. If $T(Z)$ is defined as z , then the extended transformation will also be a 1 — 1 correspondence. However, the proof of the above statement about (R_{t_i}) and the proof that the extended transformation is a homeomorphism are both long and detailed. Since they follow along lines for similar proofs it has been decided to omit them in order to conserve space. A demonstration analogous to the one needed can be found in a paper by G. T. Whyburn entitled "On the set of all cut points of a continuous curve."*

6. The converse question on the types of subset of an acyclic Peano space which can be so ordered that Axioms 0-8 are valid will be treated next.

Definition. A subset M of an acyclic Peano space is said to have *Property C'* provided that for any three distinct points a , b and x of M for which the arcs ab and ax contain a point of M different from a , there is a point p of M which lies on $ab \cdot ax$ and is such that ap is the smallest arc containing the set $ab \cdot ax \cdot M$.

In particular, it can be shown that any closed subset M has Property C' . Also, if the points of the acyclic space are ordered in terms of cut point and the points of the subset M given the same order relations as they have in the whole space, then it can be shown that Property C' implies Property C announced earlier.

Suppose then that M is a subset of an acyclic Peano space P in which betweenness has been defined in terms of cut point. If X and Y are any two points of M which are not separated in P by any point of M , then from the definition of betweenness they lie in the same cyclic element. Let $\overline{XY}$ be the arc from X to Y and Z any point which is conjugate to both X and Y . Thus, there is a point W such that the arc $\overline{WZ}$ has just the point W in common with $\overline{XY} - X - Y$. If, for each point Z of the cyclic element $C(X)$ containing X and Y such an arc $\overline{WZ}$ is chosen, then the set $H(XY) = \overline{XY}$ together with the arcs $\overline{WZ}$ will be an acyclic Peano space in which the end points are the points of $C(X)$. The theorem that the end points of such a space are totally disconnected may then be applied. Thus, by the Fréchet-Urysohn theorem mentioned earlier, each $C(X)$ satisfies Axiom 8.

A set $G(AB)$ consisting of the arc $\overline{AB}$ together with arcs $\overline{WX}$ having just the point W in common with $AB - A - B$ and just the point X in common with the cyclic chain $C(AB)$ may be formed for each $C(AB)$. Further, it may be shown that $G(AB) \cdot G(UV)$ lies in $C(AB) \cdot C(UV)$ if $C(AB) \cdot C(UV)$ consists of at most one point. Thus, any collection of cyclic

* *Fundamenta Mathematicae*, vol. 14 (1930), p. 185.

chains satisfying the conditions of Axiom 7 must satisfy the conclusion, for P is compact and locally connected.

It was seen earlier that the definition of betweenness in terms of cut point satisfies Axioms 0-5 if the space P is locally connected. It is now possible to state the following theorem:

6.1. *If P is an acyclic Peano space and M any subset of P which has Property C' , then the points of M may be so ordered that Axioms 0-8 are valid.*

Up to the present time it has not been shown that Axioms 0-8 make the transform of M have Property C' . However, it would appear that this is so.

Conclusion. It can be shown that the set of all cut points of a Peano space may be so ordered that Axioms 0-8 are true. It is only necessary to order the points of the Peano space in terms of cut point as was done earlier. Thus, the following theorem, due to G. T. Whyburn,* is a consequence of Part II:

The set of cut points of any Peano space is homeomorphic with a subset of an acyclic Peano space.

Also, if Axiom 6 is replaced by the stronger

Axiom 6'. If A , B and C are distinct then $AB \cdot BC \cdot CA \neq 0$,

it will follow that Axiom 8 is trivial. For if ABC is true and $\overline{ABC}$ false, then by Axiom 6' there is a point Z giving $\overline{AZB}$, $\overline{BZC}$ and $\overline{CZA}$. Thus, any cyclic element consists of at most two points. Also, each cyclic chain $C(AB)$ consists of just the segment AB . Hence, Axioms 0-5, 6', 7 imply that M is homeomorphic with a subset of a Peano space which is acyclic. In a paper which follows the present one an arbitrary collection N is considered the points of which are so ordered that Axioms 0-4, 6' are true. In addition to these restrictions is imposed the condition that any collection of segments such that any two of them have at most an end point in common is countable. With these conditions it is shown that a metric may be so defined for N that it is separable. Further, Axioms 5 and 7 are a consequence of the definition. Hence, such a set N may be transformed into a subset of an acyclic Peano space.

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* *Fundamenta Mathematicae*, vol. 14 (1930), p. 185.

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